§7.4. Partial Fraction Decomposition. (P.F.D)

Goal: Evaluate the integral of the following form $\int \frac{\text{Polynomial 1}}{\text{Polynomial 2}} dx$.

The DEGREE of Poly 2 (in the denominator) is either 2 or 3.

$\boxed{\text{Basic Formulas}}$:

① $\int \frac{1}{ax+b} dx = \frac{1}{a} \cdot \ln|ax+b| + C$; ② $\int \frac{1}{(ax+b)^2} dx = \frac{1}{a} \cdot \left(-\frac{1}{ax+b}\right) + C$.

③ $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + C$; ④ $\int \frac{x}{ax^2+b} dx = \frac{1}{2a} \ln|ax^2+b| + C$.

Remark: ② is based on $\int \frac{1}{x^2} dx = -\frac{1}{x}$ and u-sub: $u = ax+b$.

④. is based on u-sub: $u = ax^2+b$.

e.g. ⓪. $\int \frac{x}{3x^2-1} dx \xrightarrow[du=6x \cdot dx]{u=3x^2-1} \int \frac{1}{u} \cdot \frac{1}{6} du = \frac{1}{6} \ln|u| + C = \boxed{\frac{1}{6} \ln|3x^2-1| + C}$

P.F.D.: The method to decompose the integral into sum/Difference of several terms of ①–④.

Order 2: case 1 $\int \frac{x}{x^2-1} dx = \int \frac{x}{(x+1)(x-1)} dx$: Two DIFFERENT linear factors: $(x+1)$ and $(x-1)$.

case 2 $\int \frac{x}{(x-1)^2} dx$ Two REPEATED factors: $x-1$ and $x-1$.

Case 1:

~~eg~~ 1: Evaluate $\int \frac{5x+2}{x^2+x} dx$. Remark: $x^2+x = x(x+1)$. Two Different Factors.

(Sol)

$= \int \frac{2}{x} + \frac{3}{x+1} dx$

$\overset{\text{Formula ①}}{=} \boxed{2\ln|x| + 3\ln|x+1| + C}$.

$\boxed{\text{P.F.D. Formula}}$:

S1: $\frac{5x+2}{x^2+x} = \frac{A}{x} + \frac{B}{x+1}$.

Goal: Find A, B by CROSS-MULTIPLICATION.

S2: $5x+2 = A(x+1) + B \cdot x$

S3: Plug in $x=0$, $2 = A \cdot 1 + B \cdot 0 \Rightarrow A=2$.

$x=-1$, $-5+2 = A \cdot 0 + B \cdot (-1) \Rightarrow B=3$

S4: $\frac{5x+2}{x^2+x} = \frac{2}{x} + \frac{3}{x+1}$

Case 2:

e.g. 2 $\int_0^1 \frac{5x+2}{(x+1)^2}\,dx$

Remark: $(x+1)^2$ Repeated Factor

$\boxed{\text{P.F.D. Formula:}}$

S1: $\frac{5x+2}{(x+1)^2} = \frac{A}{x+1} + \frac{B}{(x+1)^2}$

S2: Cross-MULTI by $(x+1)^2$

$5x+2 = A(x+1) + B$ (view A, B as constants)

$5x+2 = Ax + A+B$

S3: Set up equations: $\begin{cases} A=5 \\ A+B=2 \end{cases} \Rightarrow \begin{cases} A=5 \\ B=-3 \end{cases}$

S4: $\frac{5x+2}{(x+1)^2} = \frac{5}{x+1} - \frac{3}{(x+1)^2}$ Formula ① and ②

P.F.D $\int \frac{5}{x+1} - \frac{3}{(x+1)^2}\,dx$

$= 5\int \frac{1}{x+1}dx - 3\int \frac{1}{(x+1)^2}dx$

$= 5\ln|x+1| - 3\left[-\frac{1}{x+1}\right]$

$= 5\ln|x+1| + 3\cdot\frac{1}{x+1}\Big|_0^1$

$= 5\ln 2 + 3\cdot\frac{1}{2} - (5\cdot\ln 1 + 3\cdot\frac{1}{1})$

$= \boxed{5\cdot\ln 2 - \frac{3}{2}}$

Order 3:

Case 3: $\frac{*}{(x-1)(x+1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+1} + \frac{C}{x+2}$

☆☆☆ Case 4: $\frac{*}{(x-2)\cdot(x+1)^2} = \frac{A}{x-2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$ (ww 5)

☆☆☆ Case 5: $\frac{*}{(x-2)\cdot(x^2+1)} = \frac{A}{x-2} + \frac{B}{x^2+1} + \frac{Cx}{x^2+1}$ (ww 6)

e.g. 3. $\int \frac{x+4}{2x^3-8x}\,dx$

(S15)

P.F.D: $2x^3-8x = 2x(x^2-4) = 2x\cdot(x-2)\cdot(x+2)$

$\frac{x+4}{2x^3-8x} = \frac{A}{2x} + \frac{B}{x-2} + \frac{C}{x+2}$

multiply by $2x\cdot(x-2)(x+2)$.

$x+4 = A(x-2)(x+2) + B\cdot 2x\cdot(x+2) + C\cdot 2x\cdot(x-2)$

$x=0 \Rightarrow 4 = A\cdot(-2)\cdot(2) \Rightarrow A=-1$

$x=2 \Rightarrow 6 = B\cdot 4\cdot 4 \Rightarrow B=\frac{3}{8}$

$x=-2 \Rightarrow 2 = C\cdot(-4)\cdot(-4) \Rightarrow C=\frac{1}{8}$

$\frac{x+4}{2x^3-8x} = \frac{-1}{2x} + \frac{\frac{3}{8}}{x-2} + \frac{\frac{1}{8}}{x+2}$.

P.F.D $\int -\frac{1}{2x}dx + \int \frac{\frac{3}{8}}{x-2}dx + \int \frac{\frac{1}{8}}{x+2}dx$

$= \boxed{-\frac{1}{2}\ln|x| + \frac{3}{8}\ln|x-2| + \frac{1}{8}\ln|x+2| + C}$

eg4. (ww5) $\int \frac{-1}{x\cdot(x+1)^2}\,dx$.

$$= \underbrace{\int \frac{-1}{x}\,dx + \int \frac{1}{x+1}\,dx}_{\text{Formula ①}} + \underbrace{\int \frac{1}{(x+1)^2}\,dx}_{②}$$

$$= \boxed{-\ln|x| + \ln|x+1| + \left[-\frac{1}{x+1}\right] + C}$$

$$= -\ln|x| + \ln|x+1| - \frac{1}{x+1} + C$$

P.F.D. $\frac{-1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

times $x\cdot(x+1)^2$ both sides of the equation:

$-1 = A\cdot(x+1)^2 + B\cdot(x+1)x + C\cdot x$. Expand.

$-1 = A\cdot(x^2+2x+1) + B(x^2+x) + C\cdot x$

$-1 = \underline{(A+B)}x^2 + \underline{(2A+B+C)}\cdot x + \underline{A}$

Setup Equations:

$$\begin{cases} A+B=0 \\ 2A+B+C=0 \\ A=-1 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=1 \\ C=1 \end{cases}$$

eg5. (ww6) $\int \frac{x^2+x-2}{x(x^2+2)}\,dx$.

$$= \int \frac{-1}{x}\,dx + \int \frac{2x}{x^2+2}\,dx + \int \frac{1}{x^2+2}\,dx$$

① ④ ③

$$= \boxed{-\ln|x| + \ln|x^2+2| + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C}$$

Hint: $\int \frac{2x}{x^2+2}\,dx \xrightarrow[du=2x\,dx]{u=x^2+2} \int \frac{du}{u}$

$= \ln|u|$

$= \ln|x^2+2| + C$.

$\int \frac{1}{x^2+2}\,dx = \int \frac{1}{x^2+(\sqrt{2})^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$ with $a=\sqrt{2}$ in formula ③.

P.F.D. $\frac{x^2+x-2}{x\cdot(x^2+2)} = \frac{A}{x} + \frac{Bx}{x^2+2} + \frac{C}{x^2+2}$.

multiply by $x\cdot(x^2+2)$

$1\cdot x^2 + 1x - 2 = A(x^2+2) + Bx\cdot x + C\cdot x$.

$1\cdot x^2 + 1\cdot x - 2 = (A+B)\cdot x^2 + C\cdot x + 2A$

$$\begin{cases} A+B=1 \\ C=1 \\ 2A=-2 \end{cases} \Rightarrow \begin{cases} A=-1 \\ B=2 \\ C=1 \end{cases}$$

Two Remarks for WW:

ww2: $\frac{*}{x^1\cdot(2x-1)^2\cdot(x^2+5)^4}$ the terms in P.F.D. is $1+2+4=7$.

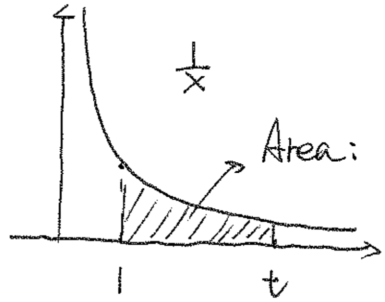
ww7: Need to use LONG-DIVISION to rewrite $\frac{5x^3-15x^2+1}{x^2-3x}$ as

$$\frac{5x\cdot(x^2-3x) + (-5x+1)}{x^2-3x} = 5x + \frac{-5x+1}{x^2-3x}.$$ Then apply P.F.D.

§7.8 Improper Integral

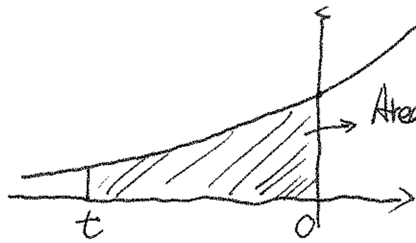
- Motivation: Evaluate the following DEFINITE integrals and explain their geometric meanings.

e.g.1 $\int_1^t \frac{1}{x}dx = \ln|x|\Big|_1^t = \ln|t| - \ln 1 = \ln|t|$ $(t>1)$



Area: $\ln|t|$ increases as t gets larger

e.g.2 $\int_t^0 e^x dx = e^x\Big|_t^0 = e^0 - e^t = 1 - e^t$ $t<0$



Area: $1-e^t$ increases as t gets smaller.

$\boxed{\text{Question:}}$ What happens to the Area (and the integral) if t goes to infinity? ($+\infty$ in eg1 , $-\infty$ in eg2)

	e.g.1	e.g.2
Area:	$\lim_{t\to+\infty} \ln\|t\| = +\infty$	$\lim_{t\to-\infty}[1-e^t] = 1-e^{-\infty} = 1$ (The limit exists and is finite)
Integral: (Notationally)	$\boxed{\int_1^{+\infty}\frac{1}{x}dx = +\infty}$	$\boxed{\int_{-\infty}^0 e^x dx = 1}$

- The above two integrals are called IMPROPER INTEGRALS. Roughly speaking, they are in the form of DEFINITE INTEGRALS, and contain some "bad points" ($\pm\infty$) in upper/lower limit of the definite integral. They represent the area of the "bad regions" (unbounded).
- The improper integral is defined as the $\boxed{\text{limit}}$ of a certain definite integral.

  If the limit exists (is some finite number), we say the improper integral $\boxed{\text{CONVERGES to that limit}}$.

  Otherwise (the limit is ∞ or doesn't exist), then we say the improper integral is $\boxed{\text{DIVERGENT}}$.
- In eg.1, $\int_1^{+\infty}\frac{1}{x}dx$ is defined as $\lim_{t\to+\infty}\int_1^t \frac{1}{x}dx$ and is $\boxed{\text{divergent}}$.

  In eg.2, $\int_{-\infty}^0 e^x dx$ is defined as $\lim_{t\to-\infty}\int_t^0 e^x dx$ and $\boxed{\text{converges to } 1}$.

e.g.3. Evaluate the improper integral $\int_0^{\infty} \frac{1}{1+4x^2}dx$ if it converges or explain why it diverges.
(S16, 12pts)

Step 1: Express the ImIn. as the limit of the definite integral. $\int_0^{+\infty} \frac{1}{1+4x^2}dx = \lim_{t\to\infty}\int_0^t \frac{1}{1+4x^2}dx$

$\boxed{\text{Step 2:}}$ Evaluate the definite integral (as an expression of t).

$$\int_0^t \frac{1}{1+4x^2}dx = \int_0^t \frac{1}{4(\frac{1}{4}+x^2)}dx$$

Hint: $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}(\frac{x}{a})+C$

$$= \frac{1}{4}\int_0^t \frac{1}{(\frac{1}{2})^2+x^2}dx$$

apply the formula with $a=\frac{1}{2}$.

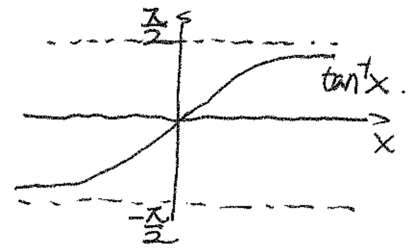
$$= \frac{1}{4}\cdot \frac{1}{\frac{1}{2}}\cdot\tan^{-1}\left(\frac{x}{\frac{1}{2}}\right)\Big|_0^t = \frac{1}{2}\tan^{-1}(2x)\Big|_0^t = \boxed{\frac{1}{2}\tan^{-1}(2t)}$$

Hint: $\tan 0 = 0$
$\Rightarrow \tan^{-1}0 = 0$.

Step 3: (Take the limit as $t\to+\infty$)

$$\lim_{t\to+\infty}\frac{1}{2}\tan^{-1}(2t) = \frac{1}{2}\tan^{-1}(+\infty) = \frac{1}{2}\cdot\frac{\pi}{2} = \boxed{\frac{\pi}{4}}$$

Hint:



Step 4: (conclusion)

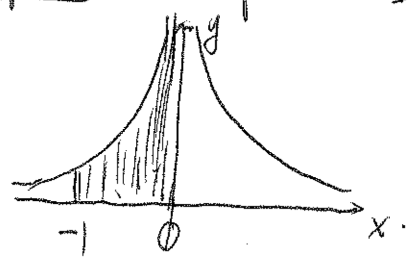
$$\boxed{\int_0^{+\infty}\frac{1}{1+4x^2}dx = \lim_{t\to\infty}\int_0^t \frac{1}{1+4x^2}dx = \lim_{t\to\infty}\frac{1}{2}\tan^{-1}(2t) = \frac{\pi}{4}}$$

i.e. $\int_0^{+\infty}\frac{1}{1+4x^2}dx$ converges to $\frac{\pi}{4}$.

- Vertical unbounded region: Improper Integral's "bad point" ~~limits~~ is a finite number.

eg.4 (f14) $\int_{-1}^{0}\frac{1}{x^2}dx$

Hint: 0 is a "bad point" since $\boxed{\frac{1}{x^2}\to+\infty \text{ as } x\to 0^-}$



$$\overset{\text{defined as}}{=} \lim_{t\to 0^-}\int_{-1}^{t}\frac{1}{x^2}dx$$

← S1: replace the 'bad point' by t and the limit of t ($\to 0^-$)

$$= \lim_{t\to 0^-}\left[-\frac{1}{x}\right]_{-1}^{t}$$

← S2: Evaluate the definite integral

$$= \lim_{t\to 0^-}\left[-\frac{1}{t}-\left(-\frac{1}{-1}\right)\right] = \lim_{t\to 0^-}\left[-\frac{1}{t}-1\right]$$

$$= -\frac{1}{0^-}-1$$

$$\boxed{= +\infty}$$

Conclusion: $\boxed{\int_{-1}^{0}\frac{1}{x^2}dx \text{ DIVERGES}}$

(More examples related to 7.1-7.4)

- eg5. $\int_1^{+\infty} \frac{-1}{x\cdot(x+1)^2}\,dx$. Determine whether the improper integral is Conv or DIV.

Step1 Step2 (Evaluate) (Refer to eg.4 in §7.4, Page 3).

$$\int_1^t \frac{-1}{x\cdot(x+1)^2}\,dx = -\ln|x| + \ln|x+1| - \frac{1}{x+1}\Big|_1^t = -\ln t + \ln(t+1) - \frac{1}{t+1} - \left[-\ln 1 + \ln 2 - \frac{1}{2}\right]$$

$$= \underbrace{-\ln t + \ln(t+1)}_{①} \underbrace{-\frac{1}{t+1}}_{②} \underbrace{-\ln 2 + \frac{1}{2}}_{③}$$

Step3: (limit as $t \to +\infty$)

②: $\lim\limits_{t\to\infty} -\frac{1}{t+1} = -\frac{1}{\infty} = 0$. All we need to worry about are the first two terms in ①.

Actually, $\lim\limits_{t\to\infty} -\ln t + \ln(t+1) = \lim\limits_{t\to\infty} \ln\frac{t+1}{t} = \ln\left[\lim\limits_{t\to\infty}\frac{t+1}{t}\right] = \ln 1 = 0$

$(-\infty + \infty)$

Step4 (Conclusion)

$$\int_1^{+\infty} \frac{-1}{x(x+1)^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{-1}{x\cdot(x+1)^2}\,dx = 0 + 0 \boxed{-\ln 2 + \frac{1}{2}}. \quad \boxed{\text{CONV}}.$$

- eg6 Evaluate $\int_1^2 \frac{dx}{x^2\sqrt{4-x^2}}$
(f15, 15pts)

Solution: Via Trig-Sub $\boxed{x = 2\sin\theta}$, $\int_1^2 \frac{dx}{x^2\cdot\sqrt{4-x^2}} = \int \frac{d\theta}{4\sin^2\theta} = \int \frac{1}{4}\csc^2\theta\, d\theta$ (Details can be found in eg.1 in §7.3, Page 7)

$$= \left[-\frac{1}{4}\cot\theta\right]\Big|_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{2}}$$

$$\int_{x=1}^{x=2} \xrightarrow{x=2\sin\theta} \int_{\sin\theta=\frac{1}{2}}^{\sin\theta=1} \longrightarrow \int_{\theta=\frac{\pi}{6}}^{\theta=\frac{\pi}{2}}$$

$$= -\frac{1}{4}\cot\frac{\pi}{2} + \frac{1}{4}\cot\frac{\pi}{6}.$$

$$\boxed{= 0 + \frac{1}{4}\cdot\sqrt{3}.}$$

(Remark: for definite integral, we do not need to solve triang)

Hints for ww.

- For some problems, we need to split the integral into TWO IMPROPER INTEGRALS. If both converge, then the original one also converges. Otherwise, it diverges.

ww4: $\int_{-\infty}^{+\infty} \frac{8}{1+36x^2}\,dx = \int_{-\infty}^{0} \frac{8}{1+36x^2}\,dx + \int_0^{+\infty} \frac{8}{1+36x^2}\,dx$.

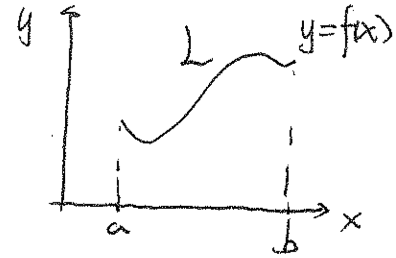
ww6: $\int_{-1}^{1} \frac{dx}{x^{8/9}} = \int_{-1}^{0} \frac{dx}{x^{8/9}} + \int_0^1 \frac{dx}{x^{8/9}}$

- Comparison Rule: If $\int_0^\infty f(x)\,dx \le \int_0^\infty g(x)\,dx$ and $\int_0^{+\infty} g(x)\,dx$ CONV, then $\int_0^{+\infty} f(x)\,dx$ also CONV. ww8,9.

§5-1 Arc length. (Remark: NOT covered in Mid 1).

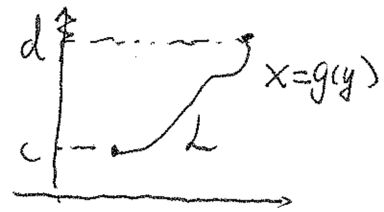
- ARC LENGTH formula of a given curve $y=f(x)$ from $x=a$ to $x=b$.

(Horizontal version): $\boxed{L=\int_a^b \sqrt{1+[f'(x)]^2}\cdot dx=\int_a^b\sqrt{1+(y')^2}\,dx}$



e.g.1 Set up an integral to find the arc length of $y=\sin x$, $0\le x\le\pi$. (Do not evaluate.)

$y'=\cos x \Rightarrow \boxed{L=\int_0^{\pi}\sqrt{1+(\cos x)^2}\cdot dx}$.



(Vertical version): $x=g(y)$, $c\le y\le d$. $\quad L=\int_c^d\sqrt{1+[g'(y)]^2}\cdot dy$

e.g.2 Find the exact arc length of $x=\frac{2}{3}\cdot y^{\frac{3}{2}}$, $0\le y\le 1$

(Vertical expression in terms of y): $L=\int_0^1\sqrt{1+[y^{\frac12}]^2}\,dy$

$x'=\frac{dx}{dy}=\frac{2}{3}\cdot\frac{3}{2}\,y^{\frac12}=y^{\frac12}$ $\quad =\int_0^1\sqrt{1+y}\cdot dy \xrightarrow[du=dy]{u=1+y}\int_1^2\sqrt{u}\,du=\frac{1}{\frac12+1}\cdot u^{\frac12+1}\Big|_{u=1}^{u=2}$

$=\frac{2}{3}\cdot u^{\frac32}\Big|_1^2$

$=\boxed{\frac{2}{3}\cdot 2^{\frac32}-\frac{2}{3}}$

e.g.3 Set up an integral to find the arc length of $y=(\ln x)^{\sqrt2}$ from the point $(e,1)$ to the point $(e^2,2^{\sqrt2})$.

(H-version): $y'=\sqrt2\cdot(\ln x)^{\sqrt2-1}\cdot\frac{1}{x}$ (chain rule)

$x=e$ to $x=e^2$ $\quad\boxed{L=\int_e^{e^2}\sqrt{1+\left[\sqrt2\cdot(\ln x)^{\sqrt2-1}\cdot\frac1x\right]^2}\,dx=\int_e^{e^2}\sqrt{1+\frac{2\cdot(\ln x)^{(\sqrt2-1)\cdot2}}{x^2}}\cdot dx}$

Hints for WW:

WW 4: use hyper-trig identity: $\cosh^2\square-\sinh^2\square=1$

WW 5: Algebra Trick: $1+\left(a-\frac{1}{4a}\right)^2=1+a^2-2\cdot a\cdot\frac{1}{4a}+\left(\frac{1}{4a}\right)^2$

(See detailed example in WW).

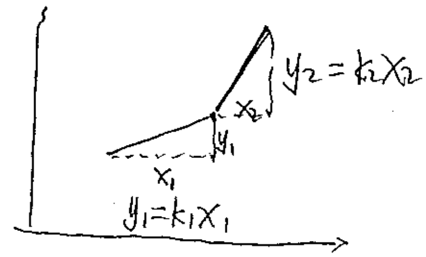
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad=1+a^2-\frac{1}{2}+\left(\frac{1}{4a}\right)^2$

$\boxed{1+\left(a-\frac{1}{4a}\right)^2 = \left(a+\frac{1}{4a}\right)^2}$

$\left(a+\frac{1}{4a}\right)^2 = a^2+2\cdot a\cdot\frac{1}{4a}+\left(\frac{1}{4a}\right)^2 \Leftarrow a^2+\frac12+\left(\frac{1}{4a}\right)^2$

Explanation of the formula:

piecewise straight line segment.



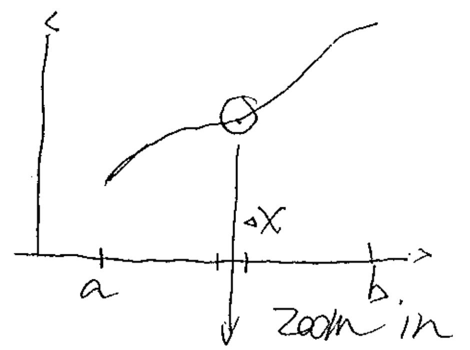
length $= \sqrt{x_1^2+y_1^2} + \sqrt{x_2^2+y_2^2}$

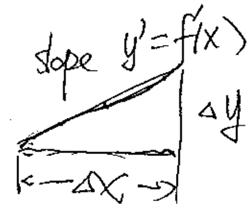
$= \sqrt{1+\left(\frac{y_1}{x_1}\right)^2} \cdot x_1 + \sqrt{1+\left(\frac{y_2}{x_2}\right)^2} \cdot x_2$.

n pieces.

Now ~~suppose~~ you have a rope $y=f(x)$

you can use several straight line segments to approach the real length





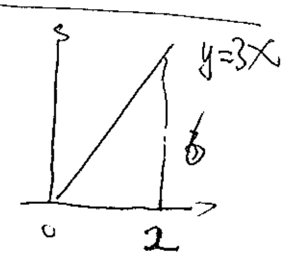
$\sqrt{\Delta x^2+\Delta y^2} = \sqrt{1+\left(\frac{\Delta y}{\Delta x}\right)^2} \cdot \Delta x$

$\overset{\Delta x \to 0}{\sim} \sqrt{1+y'^2} \cdot dx$

" ~~Sum up~~ Summation " $\longrightarrow$ Integration $\int_a^b \sqrt{1+y'^2}\, dx$.

A "trivial" example: $y=3x$, $0 \le x \le 2$

Arc length $= \sqrt{2^2+6^2} = \sqrt{40}$



or $\int_0^2 \sqrt{1+3^2}\, dx = \int_0^2 \sqrt{10} \cdot dx = 2\sqrt{10}$

(coinside)

Classification of Integrals §7.5

- Trig-Integrals (§7.2).

※13 $\int \sin^5 t\cdot\cos^4 t\cdot dt$ ☆ P5. ; ※38. $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin\theta\cdot\cot\theta}{\sec\theta}\, d\theta$ P13. ☆☆

※1. $\int \frac{\cos x}{1-\sin x}\cdot dx$ ☆+ P1 ; ※4. $\int \frac{\sin^3 x}{\cos x}\cdot dx$ ☆☆☆+ P2.

- Trig-Sub (§7.3)

※11 $\int \frac{dx}{x^3\cdot\sqrt{x^2-1}}$ ☆☆☆ P4; ※16 $\int_0^{\frac{\sqrt{2}}{2}} \frac{x^2}{\sqrt{1-x^2}}\,dx$ P6 ; ※60 $\int \frac{dx}{x^2\cdot\sqrt{4x^2-1}}$ P15.

- Partial Fractions (§7.4)

※25. $\int_0^1 \frac{1+12t}{1+3t}\,dt$ ☆ P9 ; ※9 $\int_2^4 \frac{x+2}{x^2+3x-4}\,dx$ ☆☆ P3 ; ※12 $\int \frac{2x-3}{x^3+3x}\,dx$ ☆☆☆+ P5.

- General u-Sub (in Calculus I)

☆+ ※2 $\int_0^1 (3x+1)^{\sqrt{2}}\,dx$ P1 ; ※7 $\int_{-1}^1 \frac{e^{\tan^{-1}y}}{1+y^2}\,dy$ P3 ; ※18 $\int_1^4 \frac{e^{\sqrt{t}}}{\sqrt{t}}\,dt$ P7

☆☆ ※19 $\int e^{x+e^x}\,dx$ P8 ; ※32. $\int_1^3 \frac{e^{\frac{3}{x}}}{x^2}\,dx$ P12. ☆+

☆☆☆☆ ※27 $\int \frac{dx}{1+e^x}$ P11 ; ※71 $\int \frac{e^{2x}}{1+e^x}\,dx$ P15

- IBP (§7.1)

☆☆ ※3 $\int_1^4 \sqrt{y}\cdot\ln y\,dy$ P1 ; ※15 $\int x\cdot\sec x\cdot\tan x\cdot dx$ ☆☆ P6

☆☆☆+ ※8 $\int t\sin t\cdot\cos t\cdot dt$ P3 ; ※17 $\int_0^{\pi} t\cdot\cos^2 t\cdot dt$ P7.

☆☆☆+ ※14 $\int \ln(1+x^2)\,dx$ P6 ; ※21 $\int \tan^{-1}(\sqrt{x})\,dx$. P8

- Other

※5. $\int \frac{t}{t^4+2}\,dt$ ☆☆☆ P2 ; ※6 $\int_0^1 \frac{x}{(2x+1)^3}\,dx$ ☆☆☆ P2 ; ※23 $\int_0^1 (1+\sqrt{x})^8\,dx$, P11

Classification for Improper Integrals in §7.8

★★+. (Simple linear u-Sub)

5. $\int_3^{+\infty} \frac{1}{(x-2)^{3/2}}\,dx$. P1 ; 6. $\int_0^{\infty} \frac{1}{\sqrt[4]{1+x}}\,dx$ P1 ; 7. $\int_{-\infty}^{0} \frac{1}{3-4x}\,dx$. P1

8. $\int_1^{+\infty} \frac{1}{(2x+1)^3}\,dx$ P2 ; 9. $\int_2^{+\infty} e^{-5p}\,dp$. P2 ; 28. $\int_0^{5} \frac{1}{\sqrt[3]{5-x}}\,dx$, P10

29 $\int_{-2}^{14} \frac{dx}{\sqrt[4]{x+2}}$ P11 . ★★★+ 33. $\int_0^{9} \frac{1}{\sqrt[3]{x-1}}\,dx$ P13

★★+ Direct anti-Derivative formula:

10. $\int_{-\infty}^{0} 2^r\,dr$ P2 ; 31. $\int_{-2}^{3} \frac{1}{x^4}\,dx$ P12 ; 32. $\int_0^{1} \frac{dx}{\sqrt{1-x^2}}$ P12

★★★ Typical U-Sub method.

11. $\int_0^{+\infty} \frac{x^2}{\sqrt{1+x^3}}\,dx$. P3 ; 14. $\int_1^{+\infty} \frac{e^{-\frac{1}{x}}}{x^2}\,dx$ P4 ; 16. $\int \sin\theta \cdot e^{\cos\theta}\,d\theta$ P4.

21. $\int_1^{+\infty} \frac{\ln x}{x}\,dx$ P7 24. $\int_e^{+\infty} \frac{1}{x(\ln x)^2}\,dx$. P8 ; 38 $\int \frac{\cos\theta}{\sqrt{\sin\theta}}\,d\theta$. P15

★★★+ ~ 4 Stars. IBP method.

19 $\int_{-\infty}^{0} z\cdot e^{2z}\,dz$. P6 ; 20 $\int_2^{+\infty} y\cdot e^{-3y}\,dy$ P6 ; 22 $\int_1^{+\infty} \frac{\ln x}{x^2}\,dx$ P7

37. $\int_0^1 r\cdot \ln r\,dr$ P15

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3 half ~ 4 Stars Partial Fraction

17. $\int_1^{+\infty} \frac{1}{x(x+1)}\,dx$ P5 ; 18 $\int_2^{+\infty} \frac{dv}{v^2+2v-3}$ P5 ; 34. $\int_0^{5} \frac{w}{w-2}\,dw$ P13 ; 36. $\int \frac{dx}{x^2-x-2}$ P14

Trig-Integral.

3 stars 15. $\int \sin^2\alpha\,d\alpha$ P4 4 stars+. 35. $\int \tan^2\theta\,d\theta$ P14.

Over 5 stars, super-hard. 23, 25, 26, 30, 39, 40

4 half 6 stars